Nonlinear Integrable Equations and Nonlinear Fourier Transform

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Introduction

In this paper we study nonlocal functionals whose kernels are homogeneous generalized functions. We also use such functionals to solve the Korteweg-de Vries (KdV), the nonlinear Schrödinger (NLS) and the Davey-Stewartson (DS) equations.

The solution of certain integrable equations in terms of formal power series was obtained in [4], [5]. In these papers the solution was expressed in a formal power series involving scattering data. In this paper in addition to developing techniques for multiplying and inverting nonlocal functionals we also:

(a) Give the correct version of these series by giving meaning to the relevant kernels , see (2.10) and (3.18)).

(b) We invert these series to obtain scattering data in terms of initial data.

(c) Prove the convergence of these series.

(d) We extend these results to equations in two space dimensions.

1 Nonlocal analytic functionals with homogeneous kernels

The calculus of local functionals was developed by Gelfand and Dikii [4]. Local functionals of one function $u(x)$ can be written as multiple integrals

using the kernels given by the δ-function and its derivatives. For example,

$$\int u^2(x)dx = \int u(x_1)u(x_2)\delta(x_1 - x_2)\ dx_1 dx_2\ ,$$

$$\int (u')^3 dx = \int u(x_1)u(x_2)u(x_3)\delta'(x - x_1)\delta'(x - x_2)\delta'(x - x_3)\ dx dx_1 dx_2 dx_3\ ,$$

$$\int u^2(u')^3 dx = \int u(x_1)u(x_2)u(x_3)u(x_4)u(x_5)\delta(x - x_1)\delta(x - x_2)\delta'(x - x_3)$$

$$\delta'(x - x_4)\delta'(x - x_5)\ dx dx_1 dx_2 \dots dx_5\ .$$

and so on.

Nonlocal analytic functionals are those functionals whose kernels involve homogeneous generalized functions.

In the case of a real variable a basis in the space of homogeneous generalized functions is [1]

$$\frac{{x_+}^{\lambda-1}}{\Gamma(\lambda)}, \quad \frac{{x_-}^{\lambda-1}}{\Gamma(\lambda)}, \quad (x + i0)^\lambda.$$

In the case of a complex variable a basis in the space of homogeneous generalized functions is

$$z^s \bar{z}^{s+n} \qquad n = 0, \pm 1, \pm 2, \dots$$

$$\delta - function\ \ and\ its\ derivatives$$

Remarks.

1) Only those functionals which make sense in the framework of generalized functions are allowed. For example, the functionals

$$\int u(k_1)u(k_2)\frac{1}{k_1 + k_2 + i0}\delta(k_1 + k_2)\ dk_1 dk_2,$$

and

$$\int u(k_1)u(k_2)\frac{1}{k_1 + k_2 + i0}\frac{1}{k_1 + k_2 - i0}\ dk_1 dk_2,$$

are not allowed, while the functional

$$\int u^2(k_1)u(k_2)\frac{1}{(k_1 + k_2)^2} dk_1 dk_2 := \frac{1}{2}\int_0^\infty dk \int_{-\infty}^\infty dq \frac{1}{k^2}\left(u^2\left(\frac{q + k}{2}\right) u\left(\frac{q - k}{2}\right)\right.$$

$$\left. + u^2\left(\frac{q - k}{2}\right) u\left(\frac{q + k}{2}\right) - 2u^3(q)\right)$$

is allowed. (see [1] for details).
2) Local functionals are a particular case of functionals with homogeneous kernels,for example,

$$\int u^2(x)dx = \int u(x_1)u(x_2)\delta(x_1-x_2)\, dx_1dx_2 = \int u(x_1)u(x_2)\frac{(x_1-x_2)_+^{\lambda-1}}{\Gamma(\lambda)}\bigg|_{\lambda=0} dx_1dx_2.$$

The product of two nonlocal analytic functionals is also a nonlocal analytic functional whose kernel is the direct product of the kernels of the two starting functionals. For example,

$$\left(\int u(x_1)u(x_2)\frac{(x_1-x_2)^\lambda}{\Gamma(\lambda+1)}\right)\cdot\left(\int u(y_1)u(y_2)u(y_3)\frac{(y_1-y_2)^{\mu_1}}{\Gamma(\mu_1+1)}\frac{y_3^{\mu_2}}{\Gamma(\mu_2+1)}\right)$$
$$= \int u(x_1)u(x_2)u(x_3)u(x_4)u(x_5)\frac{(x_1-x_2)^\lambda}{\Gamma(\lambda+1)}\frac{(x_3-x_4)^{\mu_1}}{\Gamma(\mu_1+1)}\frac{{x_5}^{\mu_2}}{\Gamma(\mu_2+1)}\; dx_1\dots dx_5$$

There are certain relations in the algebra of nonlocal analytic functionals.

Examples.

1)

$$\left(\int u_1(x_1)u_2(x_2)\Theta(1-x_1)\Theta(x_1-x_2)\Theta(x_2)\; dx_1dx_2\right)\cdot$$
$$\left(\int v_1(y_1)v_2(y_2)v_3(y_3)\Theta(1-y_1)\Theta(y_1-y_2)\Theta(y_2-y_3)\Theta(y_3)\; dy_1dy_2dy_3\right)$$
$$= \int_0^1 u_1(x_1)u_2(x_2)v_1(x_3)v_2(x_4)v_3(x_5)\Theta_{12}\Theta_{34}\Theta_{45}\; dx_1\dots dx_5$$
$$= \int_0^1 u_1(x_1)u_2(x_2)v_1(x_3)v_2(x_4)v_3(x_5)\,(\Theta_{12}\Theta_{23}\Theta_{34}\Theta_{45} + \Theta_{13}\Theta_{32}\Theta_{24}\Theta_{45}$$
$$+\Theta_{13}\Theta_{34}\Theta_{42}\Theta_{25} + \Theta_{13}\Theta_{34}\Theta_{45}\Theta_{52} + \Theta_{31}\Theta_{12}\Theta_{24}\Theta_{45}$$
$$+\Theta_{31}\Theta_{14}\Theta_{42}\Theta_{25} + \Theta_{31}\Theta_{14}\Theta_{45}\Theta_{52} + \Theta_{34}\Theta_{41}\Theta_{12}\Theta_{25}$$
$$+\Theta_{34}\Theta_{41}\Theta_{15}\Theta_{52} + \Theta_{34}\Theta_{45}\Theta_{51}\Theta_{12})\; dx_1\dots dx_5$$

where $\Theta(x) = \dfrac{x_+^0}{\Gamma(1)} = \begin{cases} 1, & x\geq 0 \\ 0, & x<0\end{cases}$, and $\Theta_{ij} := \Theta(x_i-x_j)$

We have multiplied two functionals, with kernels of degree 0, and with integration domain given by the simplexes, $0 < x_1 < x_2 < 1$ and $0 < y_1 < y_2 < y_3 < 1$,respectively. For the product functional, the integration domain is not a simplex, but we can write it as sum of functionals, such that the integration domain of each functional is a simplex . The simplexes are given by all possible orderings of the letters x_1, x_2, y_1, y_2, y_3 such that $x_1 < x_2$ and $y_1 < y_2 < y_3$ for all the orderings (all shuffles of $(x_1x_2)(y_1y_2y_3)$).

Remark. The functional $\int u_1(x_1)u_2(x_2)\Theta(y-x_1)\Theta(x_1-x_2)\Theta(x_2)\, dx_1dx_2$ with $u_2(x) = \frac{1}{1-x}$, $u_1(x) = \frac{1}{x}$ is the dilogarithm $Li_2(y)$.

Example 2.

$$\int u_1(x_1)u_2(x_2)u_3(x_3)u_4(x_4)u_5(x_5)\Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54}\; dx_1 \ldots dx_5$$
$$-\left(\int u_1(x_1)u_2(x_2)u_3(x_3)u_4(x_4)u_5(x_5)\Theta_{12}\Theta_{32}\; dx_1 \ldots dx_3\right)$$
$$\left(\int u_4(x_4)u_5(x_5)\; \Theta_{54}dx_4 = \int u_1(x_1)u_2(x_2)\ldots u_5(x_5)\Theta_{12}\Theta_{23}\Theta_{34}\Theta_{45}\; dx_1 \ldots dx_5\right.$$

Example 3.

$$\left(\int u_1(k_1)u_2(k_2)\frac{1}{k_1+i0}\delta(k_1+k_2)\frac{dk_1dk_2}{(2\pi i)^2}\right)$$
$$\cdot\left(\int v_1(q_1)v_2(q_2)v_3(q_3)\frac{1}{(q_1+i0)(q_1+q_2+i0)}\delta(q_1+q_2+q_3)\cdot\frac{dq_1dq_2dq_3}{(2\pi i)^3}\right)$$
$$=\int u_1(k_1)u_2(k_2)v_1(k_3)v_2(k_4)v_3(k_5)\frac{\delta(k_1+k_2)\delta(k_1+k_2+k_3)}{(k_1+i0)(k_3+i0)(k_3+k_4+i0)}\frac{dk_1\ldots dk_5}{(2\pi i)^5}$$
$$=\int u_1(k_1)u_2(k_2)v_1(k_3)v_2(k_4)v_3(k_5)\delta(k_1+k_2+k_3+k_4+k_5)$$
$$\cdot\,(p(1,2,3,4)+p(1,3,2,4)+p(1,3,4,2)+p(1,3,4,5)+p(3,1,2,4)$$
$$+p(3,1,4,2)+p(3,1,4,5)+p(3,4,1,2)+p(3,4,1,5)+p(3,4,5,1))\;\;\frac{dk_1\ldots dk_5}{(2\pi i)^4}$$

where

$$p(m_1,m_2,m_3,m_4)$$
$$=\frac{1}{(k_{m_1}+i0)(k_{m_1}+k_{m_2}+i0)(k_{m_1}+k_{m_2}+k_{m_3}+i0)(k_{m_1}+k_{m_2}+k_{m_3}+k_{m_4}+i0)}$$

Nonlocal analytic functionals appear naturally in nonlinear integrable equations. For example, in the KdV equation the transformation from the potential to scattering data and the inverse transformation are given by nonlocal analytic functionals.

We will write these functionals using the inverse scattering formalism [3]. Alternatively,one could start directly from the nonlinear equation.

2 Nonlocal functionals for the KdV equation

Let $u(x)$ be a C^∞ real-valued function of a real variable x, with the fast decrease as $x \to \pm\infty$.

We construct the following functionals of u :

$$\begin{aligned} a(k) = 1 + \sum_{n=1}^{\infty} (-)^n \int u(x_2)u(x_4)\dots u(x_{2n})\Theta_{12}\Theta_{23}\Theta_{34}\cdot\dots\cdot\Theta_{2n-1,2n} \\ \delta(x - x_1 + x_2 - \dots + x_{2n}) \exp(2ikx)\ dxdx_1dx_2\dots dx_{2n}, \end{aligned} \tag{1}$$

$$\begin{aligned} b(k) = \frac{1}{2i(k+i0)} \sum_{n=0}^{\infty} (-)^{n+1} \int u(x_1)u(x_3)\cdot\dots\cdot u(x_{2n+1})\Theta_{12}\Theta_{23}\cdot\dots\cdot\Theta_{2n,2n+1} \\ \delta(-x + x_1 - x_2 + x_3 - \dots + x_{2n+1}) \exp(-2ikx)\ dxdx_1\dots dx_{2n}, \end{aligned} \tag{2}$$

$$\begin{aligned} \Psi(k,x) = 1 + \sum_{n=1}^{\infty} (-)^n \int u(x_2)u(x_4)\cdot\dots\cdot u(x_{2n})\Theta(x_1 - x)\Theta_{21}\Theta_{32}\cdot\dots\cdot\Theta_{2n,2n-1} \\ \delta(x_0 - x_1 + x_2 - \dots + x_{2n}) \exp(-2ikx_0)\ dx_0dx_1\dots dx_{2n}, \end{aligned} \tag{3}$$

$$\begin{aligned} \Phi(k,x) = 1 + \sum_{n=1}^{\infty} (-)^n \int u(x_2)u(x_4)\cdot\dots\cdot u(x_{2n})\Theta(x - x_1)\Theta_{12}\Theta_{23}\cdot\dots\cdot\Theta_{2n-1,2n} \\ \delta(-x_0 - x_1 + x_2 - \dots + x_{2n}) \exp(-2ikx_0)\ dx_0dx_1\dots dx_{2n}. \end{aligned} \tag{4}$$

Remark. In the series, say, for $\Psi(k,x)$ we can integrate over x_{2m+1}, $m = 0, 1, \ldots$, to get another formula for $\Psi(k,x)$:

$$\Psi(k,x) = 1 + \sum_{n=1}^{\infty} \frac{(-)^n}{(2ik)^n} \int u(x_1)u(x_2)\ldots u(x_{n-1})u(x_n)$$
$$\Theta(x_1 - x)\Theta_{21}\Theta_{32}\Theta_{43} \cdot \ldots \cdot \Theta_{n,n-1}$$
$$\cdot(e^{-2ik(x-x_1)} - 1)(e^{-2ik(x_1-x_2)} - 1)\ldots(e^{-2ik(x_{n-1}-x_n)} - 1)\ dx_1 \ldots dx_n.$$

We define $S(k)$ by

$$S(k) = \frac{b(k)}{a(k)}. \tag{5}$$

If $|1 - a(k)| < 1$, $S(k)$ can be written as

$$S(k) = -\frac{1}{2i(k+i0)} \sum_{n=0}^{\infty} \int u(x_1)u(x_3)\ldots u(x_{2n+1})$$
$$\Theta_{21}\Theta_{23}\Theta_{43}\Theta_{45}\ldots\Theta_{2n,2n-1}\Theta_{2n,2n+1}$$
$$\delta(-x + x_1 - x_2 + x_3 - \ldots + x_{2n+1}) \exp(-2ikx)\ dx dx_1 dx_2 \ldots dx_{2n+1} \tag{6}$$

Convergence.
The series (1)–(4) converge for $k \neq 0$. They also converge at $k = 0$ if the moments of function $u(x)$, that is, $\int u(x)dx$, $\int xu(x)dx$, $\int x^2u(x)dx$, $\ldots$ are small. The series (5) is convergent if (1) and (2) are convergent, and $|1 - a(k)| < 1$.

Indeed, let $k \neq 0$.Then

$$|\Phi(k,x)| \leq 1 + \sum_{n=1}^{\infty} \frac{1}{|k|^n} \int |u(x_1)u(x_2)\ldots u(x_n)$$
$$\sin k(x - x_1) \sin k(x_1 - x_2) \ldots \sin k(x_{n-1} - x_n)$$
$$\Theta(x - x_1)\Theta_{12}\Theta_{23}\Theta_{34} \cdot \ldots \cdot \Theta_{n-1,n}\ dx_1 dx_2 \ldots dx_n$$
$$\leq 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \frac{(\int |u(x)|dx)^n}{k^n}$$

For all k, including $k=0$,

$$|\Phi(k,x)| \leq 1 + \sum_{n=1}^{\infty} \int |u(x_1)u(x_2)\dots u(x_n)(x-x_1)(x_1-x_2)\dots(x_{n-1}-x_n)|$$
$$\cdot\Theta(x-x_1)\Theta_{12}\cdot\ldots\cdot\Theta_{n-1,n}\; dx_1dx_2\dots dx_n$$

If the moments of function u are small, the series is convergent for $k=0$ as well.

We will consider two operations on functionals (1)–(5): multiplication and inversion. These operations are infinite-dimensional analogues of multiplication of functions and the inverse function.

There could be some relations among the products of functionals.

Example 1.

Let us show that $a(k)a(-k) - b(k)b(-k) = 1$ for all $k \neq 0$. Indeed,

$$a(k)a(-k) = 1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}(-)^n \int u(x_2)u(x_4)\dots u(x_{2n})\Theta_{12}\Theta_{23}\dots\Theta_{2m-1,2m}$$
$$\cdot\Theta_{2m+2,2m+1}\cdot\Theta_{2m+3,2m+2}\dots\Theta_{2n,2n-1}e^{2ikx_0}$$
$$\delta(-x_0+x_1-x_2+x_3-x_4+\ldots-x_{2n})\; dx_0dx_1dx_2\dots dx_{2n},$$

$$b(k)b(-k) = \sum_{n=1}^{\infty}\sum_{m=1}^{n-1}(-)^n \int u(x_2)u(x_4)\dots u(x_{2n})\Theta_{12}\Theta_{23}\cdot\ldots\cdot\Theta_{2m,2m+1}$$
$$\Theta_{2m+3,2m+2}\cdot\Theta_{2m+4,2m+3}\cdot\ldots\cdot\Theta_{2n,2n-1}e^{2ikx_0}$$
$$\delta(-x_0+x_1-x_2+x_3-x_4+\ldots-x_{2n})\; dx_0dx_1dx_2\dots dx_{2n},$$

$$a(k)a(-k) - b(k)b(-k) - 1 = \sum_{n=1}^{\infty}(-)^n \int u(x_2)u(x_4)\dots u(x_{2n})$$
$$\exp(2ik(x_1-x_2+\ldots-x_{2n}))\Theta_{23}\Theta_{34}\cdot\Theta_{45}\cdot\ldots\cdot\Theta_{2n-1,2n}$$
$$dx_1\dots dx_{2n} = 0, \quad k\neq 0.$$

From ( ) one can see that $\overline{a(k)} = a(-k), \overline{b(k)} = b(-k)$; therefore, $|a(k)| \geq 1$ and $|S(k)| = \left|\frac{b(k)}{a(k)}\right| = \frac{|b(k)|}{\sqrt{1+|b(k)|^2}} \leq 1, \quad k \neq 0$.

Example 2.

Let us show that $\Psi(-k,y)a(k) + \Psi(k,y)b(k)e^{2iky} = \Phi(k,y)$. Indeed,

$$\Psi(-k,y)a(k) = 1 + \sum_{n=1}^{\infty}(-)^n \int (u(x_2)u(x_4)\dots u(x_{2n}))$$
$$\cdot(\Theta_{12}\Theta_{23}\Theta_{34}\cdot\ldots\cdot\Theta_{2n-1,2n}$$
$$+\sum_{m=1}^{n-1}(\Theta(x_1-y)\Theta_{21}\Theta_{32}\cdot\ldots\cdot\Theta_{2m,2m-1}\Theta_{2m+1,2m+2}$$
$$\cdot\Theta_{2m+2,2m+3}\cdot\Theta_{2n-1,2n}) + \Theta(x_1-y)\Theta_{21}\Theta_{32}\cdot\ldots\cdot\Theta_{2n,2n-1})$$
$$e^{-2ikx_0}\delta(x_0+x_1-x_2+\ldots-x_{2n})\,dx_0\dots dx_{2n},$$

also,

$$\Psi(k,y)b(k)e^{2iky} = \sum_{n=1}^{\infty}(-)^{n+1}\int u(x_2)u(x_4)\cdot\ldots\cdot u(x_{2n})$$
$$\cdot(\Theta(x_1-y)\Theta_{23}\Theta_{34}\Theta_{45}\cdot\ldots\cdot\Theta_{2n-1,2n}$$
$$+\sum_{m=1}^{n-2}\Theta(x_1-y)\Theta_{21}\Theta_{32}\cdot\ldots\cdot\Theta_{2m+1,2m}\cdot\Theta_{2m+2,2m+3}$$
$$\cdot\Theta_{2m+3,2m+4}\cdot\ldots\cdot\Theta_{2n-1,2n} + \Theta(x_1-y)\Theta_{21}\Theta_{32}\cdot\ldots\cdot\Theta_{2n-1,2n})$$
$$e^{-2ikx_0}\delta(x_0+x_1-x_2+\ldots-x_{2n})\,dx_0dx_1\dots dx_{2n},$$

and

$$\Psi(-k,y)a(k) + \Psi(k,y)b(k)e^{2iky} = 1 + \sum_{n=1}^{\infty}(-)^n\int u(x_2)u(x_4)\cdot\ldots\cdot u(x_{2n})$$
$$\Theta(y-x_1)\Theta_{12}\Theta_{23}\cdot\ldots\cdot\Theta_{2n-1,2n}e^{-2ikx_0}$$
$$\delta(x+x_1-x_2-\ldots+x_{2n-1}-x_{2n})\,dx_0dx_1\dots dx_{2n} = \Phi(k,y).$$

Example 3.

Let us take $S(k)$,given by the formal series (6), and $a(k)$, $b(k)$, given by the convergent series (1), (2). We can prove the following relation for the series in u(x):

$$a(k)S(k) = b(k).$$

The computation is similar to that of the Examples 1 and 2.

Inversion.

Let us define $S(x) = \int S(k)e^{2ikx}\frac{dk}{\pi}$. We suppose that $S(x)$ is a fast decreasing function as $x \to +\infty$. Formula (6) can be written as

$$\frac{d}{dx}S(x) = -\Big(u(x) + \int u(x_1)u(x_3)\Theta_{21}\Theta_{23}\delta(x - x_1 + x_2 - x_3)dx_1dx_2dx_3$$
$$+ \ldots \int u(x_1)u(x_3)\ldots u(x_{2n+1})\Theta_{21}\Theta_{23}\Theta_{43}\Theta_{45}\ldots\Theta_{2n,2n-1}\Theta_{2n,2n+1}$$
$$\delta(x - x_1 + x_2 - x_3 + \ldots - x_4)\ dx_1dx_2\ldots dx_{2n+1} + \ldots\Big)$$

(here the right-hand side is a formal series in $u(x)$).

We can invert it, that is, we can express $u(x)$ in terms of $S(x)$ by the formal series $u(x) = \sum_{k=1}^{\infty} W_k(x)$, where $W_k(x)$ is a nonlocal analytic functional of degree k in $S(x)$ ($S(x)$ has degree one). The functionals $W_k(x)$ are determined recursively:

$$W_1(x) = -\frac{d}{dx}S(x),$$
$$W_2(x) = -\int \frac{d}{dx_1}S(x_1)\frac{d}{dx_3}S(x_3)\Theta_{21}\Theta_{23}\delta(x - x_1 + x_2 - x_3)\ dx_1dx_2dx_3$$
$$= -\int S(x_1)S(x_3)\,(\delta(x_2 - x_1)\delta(x_2 - x_3)\delta(x - x_1 + x_2 - x_3)$$
$$+\delta(x_2 - x_1)\Theta_{23}\delta'(x - x_1 + x_2 - x_3)$$
$$+\Theta_{21}\delta(x_2 - x_3)\delta'(x - x_1 + x_2 - x_3)$$
$$+\Theta_{23}\Theta_{23}\delta''(x - x_1 + x_2 - x_3))\ \ dx_1dx_2dx_3$$

$$= -\int S(x_1)S(x_3)\delta(x_2 - x_1)\delta(x_2 - x_3)\delta(x - x_1 + x_2 - x_3)\ dx_1dx_2dx_3$$

$$= -S^2(x) = \frac{d}{dx}\int S(x_1)S(x_2)\Theta(x_1 - x)\delta(x_1 - x_2)\ dx_1dx_2,$$

(we integrated by parts),

$$W_n(x) = -\sum_{k=2}^{n} \sum_{\substack{m_1, m_2 \ldots m_k \geq 1 \\ m_1 + m_2 + \ldots + m_k = n}} \int W_{m_1}(x_1)W_{m_2}(x_3)\ldots W_{m_k}(x_{2k+1})$$
$$\Theta_{21}\Theta_{23}\cdot\Theta_{43}\Theta_{45}\cdot\ldots\cdot\Theta_{2k,2k-1}\Theta_{2k,2k+1}$$
$$\delta(x - x_1 + x_2 - \ldots + x_{2k} - x_{2k+1})\ dx_1dx_2\ldots dx_{2k+1}.$$

Lemma. *Consider the functionals of $S(x)$, given by the formal series*

$$\begin{aligned}
\tilde{u}(x) = &-\frac{d}{dx}S(x) + \frac{d}{dx}\int S(x_1)S(x_2)\Theta(x_1 - x)\delta(x_1 - x_2)\ dx_1dx_2 \\
&-\frac{d}{dx}\sum_{n=1}^{\infty}\int S(x_1)S(x_2)\ldots S(x_{2n+1})\Theta(-x_1 + x_2) \\
&\quad \Theta(-x_1 + x_2 - x_3 + x_4)\ldots\Theta(-x_1 + x_2 - \ldots + x_{2n})\Theta(-x_{2n+1} + x_{2n}) \\
&\quad \Theta(-x_{2n+1} + x_{2n} - x_{2n-1} + x_{2n-2})\ldots\Theta(-x_{2n+1} + x_{2n} - \ldots + x_2) \\
&\quad \delta(x - x_1 + x_2 - x_3 + \ldots - x_{2n+1})\ dx_1dx_2\ldots dx_{2n+1} \\
&+\frac{d}{dx}\sum_{n=2}^{\infty}\int S(x_1)S(x_2)\ldots S(x_{2n}) \\
&\quad \Theta(-x_1 + x_2)\Theta(-x_1 + x_2 - x_3 + x_4)\ldots\Theta(-x_1 + x_2 - \ldots + x_{2n-2}) \\
&\quad \Theta(x_1 - x)\Theta(x_1 - x_2 + x_3 - x)\ldots\Theta(x_1 - x_2 + x_3 - \ldots + x_{2n-1} - x) \\
&\quad \delta(x - x_2 + x_3 - x_4 + \ldots - x_{2n})\ dx_1\ldots dx_{2n}.
\end{aligned} \tag{7}$$

$$
\begin{aligned}
\tilde{\Psi}(x,y) = \delta(x) - \sum_{n=0}^{\infty} \Theta(-x) \int S(x_1)S(x_3)\dots S(x_{2n+1}) \\
\Theta(-x_1+x_2)\Theta(-x_1+x_2-x_3+x_4)\dots\Theta(-x_1+x_2-\dots+x_{2n}) \\
\Theta(x_1-y)\Theta(x_1-x_2+x_3-y)\dots\Theta(x_1-x_2+x_3-\dots+x_{2n+1}-y) \\
\delta(x-y+x_1-x_2+x_3-\dots+x_{2n+1})\ dx_1\dots dx_{2n+1} \\
+\sum_{n=1}^{\infty} \Theta(-x) \int S(x_1)S(x_2)\dots S(x_{2n}) \\
\Theta(-x_1+x_2)\Theta(-x_1+x_2-x_3+x_4)\Theta(-x_1+x_2-\dots+x_{2n-2}) \\
\Theta(x_1-y)\Theta(x_1-x_2+x_3-y)\dots\Theta(x_1-x_2+x_3-\dots+x_{2n-1}-y) \\
\delta(x-x_1+x_2-\dots+x_{2n})\ dx_1dx_2\dots dx_{2n}.
\end{aligned}
\tag{8}
$$

We can substitute in these series $S(x)$ *as a functional of* $\{u(x)\}$ *, given by the formal series*

$$
\begin{aligned}
S(x) = \int \Theta(x_1-x)u(x_1)dx_1 + \sum_{n=1}^{\infty} \int u(x_1)u(x_3)\dots u(x_{2n+1}) \\
\Theta_{21}\Theta_{23}\Theta_{43}\Theta_{45}\cdot\ldots\cdot\Theta_{2n,2n-1}\Theta_{2n,2n+1} \\
\cdot\delta(x_0-x_1+x_2-x_3+\dots-x_{2n+1})\ dx_0dx_1\dots dx_{2n+1}.
\end{aligned}
\tag{9}
$$

As a result of this substitution, we will have $\tilde{u}(x)$ *and* $\tilde{\Psi}(x,y)$ *given by formal series in* $\{u(x)\}$*. Moreover,*

$\tilde{u}(x) = u(x)$,

$\tilde{\Psi}(x,y) = \Psi(x,y) := \int \Psi(k,y)e^{2ikx}\frac{dk}{\pi}$, *where* $\Psi(k,y)$ *is given by* (3)*.*

Proof

We will prove the lemma by induction in the degree of $\{u(x)\}$.

1) In the first order in $\{u(x)\}$

$$S_{(1)}(x) = \int \Theta(x_1 - x)u(x_1)\ dx_1,$$

$$\tilde{u}(x)_1 = -\frac{d}{dx}S_{(1)}(x) = u(x),$$

$$\tilde{\Psi}_{(1)}(x,y) = -\Theta(-x)\int S_{(1)}(x_1)\delta(x - y + x_1)\ dx_1$$

$$= -\Theta(-x)\int \Theta(x_2 - x_1)u(x_2)\delta(x - y + x_1)\ dx_1dx_2$$

$$= -\int \Theta(x_1 - y)\Theta(x_2 - x_1)u(x_2)\delta(x - y + x_1)\ dx_1dx_2$$

$$-\int \Theta(x_1 - y)\Theta(x_2 - x_1)u(x_2)\delta(x - x_1 + x_2)$$

$$\cdot dx_1dx_2 = \Psi_{(1)}(x,y).$$

(In the last step we have made the change of variables $x_1 \to y + x_2 - x_1$).

2) Suppose that we have proved that

$$\tilde{\Psi}(x,y) = \delta(x) + \sum_{n=1}^{N}(-)^n \int u(x_2)u(x_4)\ldots u(x_{2n})\Theta(x_1 - y)\Theta_{21}\Theta_{32}\ldots\Theta_{2n,2n-1}$$
$$\delta(x - x_1 + x_2 - \ldots + x_{2n})\ dx_1dx_2\ldots dx_{2n} + O(u^{N+1}).$$

From the definition of $\tilde{\Psi}(x,y)$

$$\tilde{\Psi}(x,y) = \delta(x) - \Theta(-x)\int \tilde{\Psi}(x_1,y)S(y - x - x_1)\ dx.$$

But $S(x)$ is a series in $\{u(x)\}$ with terms of degree ≥ 1 in u, therefore, if we know $\tilde{\Psi}(x,y)$ as functionals of $\{u(x)\}$ up to degree n, we can compute it in the next order $(n+1)$.

Notice that

$$(-)^n \int u(x_2)u(x_4)\ldots u(x_{2n)}\Theta(x_1-y)\Theta_{21}\Theta_{32}\Theta_{43}\ldots\Theta_{2n,2n-1}$$

$$\delta(x-x_1+x_2-\ldots x_{2n})\ dx_1dx_2\ldots+dx_{2n}$$

$$=(-)^n \int u(x_2)u(x_4)\ldots u(x_{2n})\Theta(x_1-y)\Theta_{21}\Theta_{32}\Theta_{43}\ldots\Theta_{2n,2n-1}$$

$$\delta(x-y+x_1-x_2+x_3-\ldots+x_{2n-1})\ dx_1dx_2\ldots dx_{2n}$$

$$=(-)^{n+1}\frac{d}{dx}\int u(x_2)u(x_4)\ldots u(x_{2n})\Theta(x_1-y)\Theta_{21}\Theta_{32}\Theta_{43}\ldots\Theta_{2n,2n-1}\Theta_{2n+1,2n}$$

$$\cdot\delta\left(x-y+x_1-x_2+x_3-\ldots+x_{2n-1}-x_{2n}+x_{2n+1}\right)\ dx_1dx_2dx_3\ldots dx_{2n}dx_{2n+1}$$

(In the first step we have used the change of variables $x_1 \to x_2+y-x_1, x_3 \to x_2+x_4-x_3, \ldots, x_{2n-1} \to x_{2n-2}+x_{2n}-x_{2n-1}$).

Also, $\delta(x) = -\frac{d}{dx}\int \Theta(x_1-y)\delta(x-y+x_1)dx_1$, and

$$\tilde{\Psi}_{N+1}(x,y) = -\sum_{m=0}^{N}\Theta(-x)\int \tilde{\Psi}_{(m)}(x_0,y)S_{(N+1-m)}(y-x-x_0)\ dx_0$$

$$=-\sum_{m=0}^{N}\Theta(-x)(-)^{m+1}\int\frac{d}{dx_0}\left(u(x_2)u(x_4)\ldots u(x_{2m})\right.$$

$$\Theta(x,y_1)\Theta_{21}\Theta_{32}\Theta_{43}\ldots\Theta_{2m,2m-1}\Theta_{2m+1,2m}$$

$$\cdot\delta\left(x_0-y+x_1-x_2+x_3-\ldots+x_{2m+1}\right))$$

$$\cdot S_{(N+1-m)}(y-x-x_0)dx_0dx_1dx_2\ldots dx_{2m+1}$$

$$=\Theta(-x)\int u(x_2)u(x_4)\ldots u(x_{2n+2})$$

$$\Big(\sum_{m=0}^{N}(-)^{m+1}\Theta(x_1-y)\Theta_{21}\Theta_{32}\ldots\Theta_{2m,2m-1}\Theta_{2m+1,2m}\cdot(\Theta_{2m+2,2m+1}+\Theta_{2m+1,2m+2})$$

$$(\Theta_{2m+3,2m+2}\Theta_{2m+3,2m+4}\cdot\ldots\cdot\Theta_{2N+1,2N}\Theta_{2N+1,2N+2})\Big)$$

$$\cdot\delta(-x+x_1-x_2+x_3-\ldots+x_{2N+1}-x_{2N+2})\ dx_1dx_2\ldots dx_{2N+2}$$

$$=(-)^{N+1}\Theta(-x)\int u(x_2)u(x_4)\ldots u(x_{2N+2})$$

$$\Big(\Theta(x_1-y_1)\Theta_{21}\Theta_{32}\Theta_{43}\cdot\ldots\cdot\Theta_{2N+2,2N+1}-(-)^{N+1}$$

$$\cdot\,\Theta(x_1-y)\Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54}\cdot\ldots\cdot\Theta_{2N+1,2N}\Theta_{2N+1,2N+2})$$

$$\delta(-x+x_1-x_2+x_3-\ldots+x_{2N+1}-x_{2N+2})\ dx_1dx_2\ldots x_{2N+2}$$

The second term in the last expression is zero, because the volume of the integration domain vanishes ; the first term coincides with the term of degree $(N+1)$ in $\Psi(x,y)$, see (3).

To prove the formula for $\tilde{u}(x)$ we use the relation $\tilde{u}(x) = -\frac{d}{dx}\int S(x-x_1)\tilde{\Psi}(x_1,x)dx_1$, which follows from the definition of $\tilde{u}$ and $\tilde{\Psi}$. We know both $S(x)$ and $\tilde{\Psi}(x,y)$ as functionals in $\{u(x)\}$. The calculation of the same type as above gives that only the first order term in $\{u(x)\}$ is not zero:

$$\tilde{u}(x)=\frac{d}{dx}\sum_{n=1}^{\infty}(-)^n\int u(x_2)u(x_4)\ldots u(x_{2n})\delta(-x_1+x_2-\ldots+x_{2n})$$

$$\cdot\,(\Theta(x_1-x)\Theta_{21}\Theta_{32}\Theta_{43}\cdot\ldots\cdot\Theta_{2n,2n-1}-(-)^n$$

$$\Theta(x_1-x)\Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54}\cdot\ldots\cdot\Theta_{2n-1,2n-2}\Theta_{2n-1,2n})$$

$$dx_1dx_2\ldots dx_{2n}=u(x)$$

The formula (7) can be written as follows:

$$u(x)=4\sum_{n=1}^{\infty}\int\frac{S(k_1)S(k_2)\ldots S(k_n)(k_1+k_2+\ldots+k_n)}{(k_1+k_2+i0)(k_2+k_3+i0)\ldots(k_{n-1}+k_n+i0)}$$
$$\exp(2ikx)\delta(k-k_1-k_2-\ldots-k_n)\ \frac{dkdk_1dk_2\ldots dk_n}{(2\pi i)^n} \tag{10}$$

Any polynomial in $u(x)$ and its derivative can be written as the functional of the same type, but with some polynomial in $\{k_i\}$ in the numerator. This

property is similar to the usual Fourier transform of the linear function of u and its derivatives . Therefore it is natural to call transformation (10) the Nonlinear Fourier transform of u :

$$u^{(d_1)}u^{(d_2)}\dots u^{(d_m)} = 4^m(2i)^{d_1+d_2+\dots+d_m}$$

$$\sum_{n=m}^{\infty}\int \frac{S(k_1)S(k_2)\dots S(k_n)}{(k_1+k_2+i0)(k_2+k_3+i0)\dots(k_{n-1}+k_n+i0)}$$

$$\exp(2ikx)\cdot\delta(k-k_1-k_2-\dots-k_n)\cdot \mathrm{Sym}_{d_1,d_2\dots d_m}$$

$$\sum_{1\le p_1<p_2<\dots<p_{m-1}<n}(k_1+k_2+\dots+k_{p_1})^{d_1+1}(k_{p_1}+k_{p_1+1})$$

$$(k_{p_1+1}+k_{p_1+2}+\dots+k_{p_2})^{d_2+1}(k_{p_2}+k_{p_2+1})(k_{p_2+1}+k_{p_2+2}+\dots+k_{p_3})^{d_3+1}$$

$$\cdot(k_{p_3}+k_{p_3+1})(k_{p_{m-1}+1}+k_{p_{m-1}+2}+\dots+k_n)^{d_m+1}\frac{dkdk_1dk_2\dots dk_n}{(2\pi i)^n}. \tag{11}$$

Examples.

$$6uu_x = 32i\sum_{n=2}^{\infty}\int \frac{S(k_1)S(k_2)S(k_3)\dots S(k_n)}{(k_1+k_2+i0)(k_2+k_3+i0)\dots(k_{n-1}+k_n+i0)}$$

$$\exp(2ikx)\cdot\delta(k-k_1-k_2-\dots-k_n)$$

$$\cdot k\cdot\Big((k_1+k_2+\dots+k_n)^3-({k_1}^3+{k_2}^3+\dots+{k_n}^3)\Big)\ \frac{dk\ dk_1\ dk_2\dots dk_n}{(2\pi i)^n}$$

$$u_{xxx}+6uu_x = -32i\sum_{n=1}^{\infty}\int \frac{S(k_1)S(k_2)S(k_3)\dots S(k_n)}{(k_1+k_2+i0)(k_2+k_3+i0)\dots(k_{n-1}+k_n+i0)}$$

$$\exp(2ikx)\cdot\delta(k-k_1-k_2-\dots-k_n)$$

$$\cdot k\cdot\Big({k_1}^3+{k_2}^3+\dots+{k_n}^3)\Big)\ \frac{dkdk_1dk_2\dots dk_n}{(2\pi i)^n} \tag{12}$$

We see that for the differential polynomial $u_{xxx}+6uu_x$ the corresponding polynomial in $\{k\}$ after the nonlinear Fourier transform is $({k_1}^3+{k_2}^3+\dots+{k_n}^3)$, n= 1,2, $\dots$.

We can use this fact to solve some nonlinear equation. Let $S(k,t) = S_0(k)e^{8ik^3t}$, and $u(x,t)$ is defined as a functional of $S(k,t)$ by (10) (with $S(k,t)$ instead of $S(k)$). Such $u(x,t)$ solves the KdV equation

$$-\frac{d}{dt}u(x,t) = \frac{\partial^3}{\partial x^3}u(x,t) + 6u\frac{\partial u}{\partial x}.$$

Indeed, for such $u(x,t)$ the right-hand side of the equation is expressed in terms of $S(k,t)$ by (12), and it coincides with $-\frac{d}{dt}u(x,t)$.

Let us find polynomials in u and derivatives of u such that after the Nonlinear Fourier Transform the corresponding polynomial in k is given by $\sum_i {k_i}^{2l-1}$, $l = 1, 2, \ldots$ .In order to do this, let us first compute the resolvent:

$$\Psi(k,x) = 1 + \sum_{n=1}^{\infty} \int \frac{S(k_1)S(k_2)S(k_3)\ldots S(k_n)}{(k+k_1+i0)(k_2+k_3+i0)\ldots(k_{n-1}+k_n+i0)}$$
$$\exp 2i(k_1+\ldots+k_n)x \ \frac{dk_1\ldots dk_n}{(2\pi i)^n},$$

$$R(k,x) = \Psi(k,x)\Psi(-k,x-0)$$
$$= 1 + \sum_{n=1}^{\infty} \int S(k_1)S(k_2)\ldots S(k_n)\exp(2i(k_1+\ldots+k_n)x) \cdot \Bigg(\sum_{m=1}^{n-1}$$
$$\frac{1}{(k_1+k_2+i0)(k_2+k_3+i0)\ldots(k_{m-1}+k_m+i0)(k_m+k+i0)(k_{m+1}-k+i0)(k_{m+1}+k_{m+2}+i0)\ldots(k_{n-1}+k_n+i0)}$$
$$+\frac{1}{(k+k_1+i0)(k_1+k_2+i0)\ldots(k_{n-1}+k_n+i0)}$$
$$+\left.\frac{1}{(-k+k_1+i0)(k_1+k_2+i0)\ldots(k_{n-1}+k_n+i0)}\right)\frac{dk_1\ldots dk_n}{(2\pi i)^n}$$
$$= 1 + \sum_{n=1}^{\infty} \int \frac{S(k_1)S(k_2)\ldots S(k_n)\exp(2i(k_1+\ldots k_n)x)}{(k_1+k_2+i0)(k_2+k_3+i0)\ldots(k_{n-1}+k_n+i0)}$$
$$\left(\sum_{m=1}^{n-1}\left(\frac{1}{k_m+k+i0} + \frac{1}{k_{m+1}-k+i0}\right) + \frac{1}{k+k_1+i0} + \frac{1}{-k+k_1+i0}\right)\frac{dk_1\ldots dk_n}{(2\pi i)^n}$$
$$\sim 1 - 2\sum_{n=1}^{\infty} \int \frac{S(k_1)S(k_2)\ldots S(k_n)}{(k_1+k_2+i0)(k_2+k_3+i0)\ldots(k_{n-1}+k_n+i0)} \exp 2i(k_1+\ldots+k_n)x$$
$$\sum_{l=1}^{\infty} \frac{1}{k^{2l}}(k_1^{2l-1}+k_2^{2l-1}+\ldots+k_n^{2l-1}) \cdot \frac{dk_1\ldots dk_n}{(2\pi i)^n}.$$

Lemma.

1)

$$I_1 = \frac{1}{4}\int \ u(x)dx$$

$$= \sum_{n=1}^{\infty}\int \frac{S(k_1)S(k_2)\ldots S(k_n)k\exp(2ikx)}{(k_1+k_2+i0)(k_2+k_3+i0)\ldots(k_{n-1}+k_n+i0)}$$

$$\delta(-k+k_1+\ldots+k_n)\ \frac{dkdk_1\ldots dk_n}{(2\pi i)^n}\ dx$$

$$I_3 = -\frac{1}{16}\int(3u^2+u'')$$

$$= \pi\sum_{n=1}^{\infty}\int \frac{S(k_1)\ldots S(k_n)({k_1}^3+{k_2}^3+\ldots+{k_n}^3)}{(k_1+k_2+i0)(k_2+k_3+i0)\ldots(k_{n-1}+k_n+i0)}$$

$$\delta(k_1+\ldots+k_n)\ \frac{dk_1\ldots dk_n}{(2\pi i)^n}$$

$$I_5 = \frac{1}{64}\int(10u^3+10uu''+5(u')^2+u^{1v})$$

$$= \pi\sum_{n=1}^{\infty}\int \frac{S(k_1)\ldots S(k_n)({k_1}^5+{k_2}^5+\ldots+{k_n}^5)}{(k_1+k_2+i0)(k_2+k_3+i0)\ldots(k_{n-1}+k_n+i0)}$$

$$\delta(k_1+\ldots+k_n)\ \frac{dk_1\ldots dk_n}{(2\pi i)^n}$$

$$\ldots$$

2) *let* $S(k,t) = S(k)e^{ik^{2l+1}t}, l = 0,1,2,\ldots,$ *and*

$$I_{l,m} = -\frac{1}{2}\sum_{n=1}^{\infty}\int \frac{S(k_1,t)S(k_2,t)\ldots S(k_n,t)}{(k_1+k_2+i0)(k_2+k_3+i0)\ldots(k_{n-1}+k_n+i0)}$$

$$({k_1}^{2m+1}+{k_2}^{2m+1}+\ldots+{k_n}^{2m+1})\delta(k_1+\ldots+k_n)\ \frac{dk_1\ldots dk_n}{(2\pi i)^{n-1}}$$

Such functionals $I_{l,m}$ *are conserved:* $\frac{d}{dt}I_{l,m} = 0.$

Theorem 2.1 (Solution of the KdV equation).
Consider the Cauchy problem for the KdV equation

$$-\frac{\partial}{\partial t}u(x,t) = \frac{\partial^3}{\partial x^3}u(x,t) + 6u\frac{\partial u}{\partial x}, t \geq 0 \quad u(x,0) = u(x) \tag{13}$$

(**Solution in formal power series**).

$$u(x,t) = 4\sum_{n=1}^{\infty}\int \frac{S(k_1,t)S(k_2,t)\ldots S(k_n,t)(k_1+\ldots+k_n)}{(k_1+k_2+i0)(k_2+k_3+i0)\ldots(k_{n-1}+k_n+i0)}$$
$$\exp(2i(k_1+\ldots+k_n)x)\ \frac{dk_1\ldots dk_n}{(2\pi i)^n}$$

where

$$S(k,t) = S(k)\exp(8ik^3t)$$

$$S(k) = -\frac{1}{2i(k+i0)}\sum_{n=0}^{\infty}\int u(x_1)u(x_3)\ldots u(x_{2n+1})$$
$$\Theta_{21}\Theta_{23}\Theta_{43}\Theta_{45}\cdot\ldots\cdot\Theta_{2n,2n-1}\Theta_{2n,2n+1}$$
$$\delta(-x+x_1-x_2+x_3-\ldots+x_{2n+1})\exp(-2ikx)\ dxdx_1\ldots dx_{2n+1}$$

$$u(x)\equiv u(x,0)$$

is the solution of Cauchy problem.
(**Solution in convergent series**).
1) Starting from $u(x)$ define $S(k)$, $\{i\kappa_n\}_{n=1}^N$, $\{c_n\}_{n=1}^N$ as follows:
$S(k) = \dfrac{b(k)}{a(k)}$, $k\in\mathbb{R}$, where

$$a(k) = 1+\sum_{n=1}^{\infty}(-)^n\int u(x_2)u(x_4)\ldots u(x_{2n})\exp(+2ikx_0)$$
$$\delta(x_0-x_1+x_2-\ldots+x_{2n})\Theta_{12}\Theta_{23}\ldots\Theta_{2n-1,2n}$$
$$dx_0dx_1\ldots dx_{2n},\quad k\in\mathbb{C}^+$$

$$b(k) = \frac{1}{2i(k+i0)}\sum_{n=0}^{\infty}(-)^{n+1}\int u(x_1)u(x_3)\ldots u(x_{2n+1})\exp(-2ikx_0)$$
$$\delta(-x_0+x_1-x_2+x_3-\ldots+x_{2n+1})\Theta_{12}\Theta_{23}\ldots\Theta_{2n,2n+1}$$
$$dx_0dx_1\ldots dx_{2n+1}$$

$S(k)=\bar{S}(-k); |S(k)|\leq 1, k\neq 0$

$\{i\kappa_n\}_{n=1}^N, \kappa_n \in \mathbb{R}^+$ is the set of zeroes of the function $a(k), k \in \mathbb{C}^+$

$c_n = \dfrac{\tilde{c}_n}{\frac{\partial}{\partial k}a(k)\mid_{k=i\kappa_n}}$, where

$\tilde{c}_n = \dfrac{e^{\kappa_n x}\Phi(i\kappa_n, x)}{\Psi(i\kappa_n, x)} = \dfrac{\Phi(i\kappa_n, 0)}{\Psi(i\kappa_n, 0)}$ (the ratio doesn't depend on x)

$\Phi(k, x)$ and $\Psi(k, x)$ defined in (3), (4).

Define

$$S(x, t) = \int S(k)\exp(2ikx + 8ik^3t)\frac{dk}{\pi}$$

$$c_n(t) = c_n \exp(8{\kappa_n}^3 t)$$

For the class of initial data $u(x)$ such that the function $S(x)$ is of fast decrease as $x \to +\infty$, the solution could be written as follows:

Define the Fredholm determinant and the 1st minor to be

$$D_{x,t} = 1 + \sum_{n=1}^{\infty} \frac{1}{n!}\int_{-\infty}^{\infty} \Theta(-y_1)\Theta(-y_2)\ldots\Theta(-y_n)\ldots \det_{ij}[S(x - y_i - y_j, t)]dy_1dy_2\ldots dy_n$$

$$D_{x,t}\begin{pmatrix} y \\ y_0 \end{pmatrix} = \Theta(-y)S(x - y - y_0, t) + \sum_{n=1}^{\infty}\frac{1}{n!}\Theta(-y)\int \Theta(-y_1)\Theta(-y_2)\ldots\Theta(-y_n)$$

$$\cdot \begin{vmatrix} S(x-y-y_0,t) & S(x-y-y_1,t) & \ldots & S(x-y-y_n,t) \\ S(x-y_1-y_0,t) & S(x-y_1-y_1,t) & \ldots & S(x-y_1-y_n,t) \\ & \ldots & & \\ S(x-y_n-y_0,t) & S(x-y_n-y_1,t) & \ldots & S(x-y_n-y_n,t) \end{vmatrix} \cdot dy_1dy_2\ldots dy_n$$

$$u(x, t) = -\frac{\partial}{\partial x}\int S(x - y, t)\left(\delta(y) - \frac{D_{x,t}\begin{pmatrix} y \\ 0 \end{pmatrix}}{D_{x,t}}\right)dy \quad - 2\frac{\partial^2}{\partial x^2}\ln\det A(x, t)$$

$$A(x, t)_{mn} = \delta_{mn} - \frac{ic_n(t)e^{-(\kappa_n+\kappa_m)x}}{(\kappa_n + \kappa_m)} \quad +$$

$$2ic_n(t)e^{-(\kappa_n+\kappa_m)x}\int e^{2(\kappa_m y_0 + \kappa_n y_1)}\frac{D_{x,t}\begin{pmatrix} y_0 \\ y_1 \end{pmatrix}}{D_{x,t}}\Theta(-y_1)dy_0dy_1.$$

This solution is defined for the class of initial data such that the Fredholm determinant and 1st minor are convergent.For convergence it is enough to have $S(x)$ of fast decrease as $x \to +\infty$. It can be proved that the Fredholm determinant is not zero.

3 Nonlocal transformations for the Nonlinear Schrodinger equation (defocusing case)

Let $q(x)$ be a C^∞ complex-valued function of a real variable x, with fast decrease as $x \to \pm\infty$ (Schwarz class).

From $q(x)$ construct the following series:

$$a(x) = \delta(x) + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} \bar{q}(x_1) q(x_2) \bar{q}(x_3) \ldots q(x_{2n}) \Theta_{21} \Theta_{32} \Theta_{43} \ldots \Theta_{2n,2n-1}$$
$$\delta(x + x_1 - x_2 + x_3 - \ldots + x_{2n})\; dx_1 dx_2 \ldots dx_{2n} \tag{1}$$

$$b(x) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} q(x_1) \bar{q}(x_2) q(x_3) \ldots \bar{q}(x_{2n}) q(x_{2n+1}) \Theta_{21} \Theta_{32} \Theta_{43} \cdot \ldots \cdot \Theta_{2n+1,2n}$$
$$\delta(x - x_1 + x_2 - \ldots + x_{2n} - x_{2n+1})\; dx_1 dx_2 \ldots dx_{2n+1} \tag{2}$$

$$\Phi_1(x,y) = \delta(x) + \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} q(x_1) \bar{q}(x_2) q(x_3) \ldots \bar{q}(x_{2n}) \Theta(y - x_1) \Theta_{12} \Theta_{23} \ldots \Theta_{2n+1,2n}$$
$$\cdot \delta(x - x_1 + x_2 - x_3 + \ldots - x_{2n-1} + x_{2n})\; dx_1 \ldots dx_{2n} \tag{3}$$

$$\Phi_2(x,y) = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \bar{q}(x_1) q(x_2) \bar{q}(x_3) \ldots q(x_{2n}) \bar{q}(x_{2n+1}) \Theta(y - x_1) \Theta_{12} \Theta_{23} \ldots \Theta_{2n,2n+1}$$
$$\cdot \delta(x - y + x_1 - x_2 + \ldots - x_{2n} + x_{2n+1})\; dx_1 \ldots dx_{2n+1} \tag{4}$$

$$\Psi_1(x,y) = -\sum_{n=0}^{\infty}\int_{-\infty}^{\infty} q(x_1)\bar{q}(x_2)q(x_3)\ldots\bar{q}(x_{2n})q(x_{2n+1})\Theta(x_1-y)\Theta_{21}\Theta_{32}\ldots\Theta_{2n+1,2n}$$
$$\delta(x+y-x_1+x_2-x_3+\ldots-x_{2n+1})\ dx_1\ldots dx_{2n+1} \tag{5}$$

$$\Psi_2(x,y) = \delta(x) + \sum_{n=1}^{\infty}\int_{-\infty}^{\infty} \bar{q}(x_1)q(x_2)\bar{q}(x_3)\ldots q(x_{2n})\Theta(x_1-y)\Theta_{21}\Theta_{32}\ldots\Theta_{2n,2n-1}$$
$$\delta(x+x_1-x_2+x_3-\ldots+x_{2n-1}-x_{2n})\ dx_1\ldots dx_{2n} \tag{6}$$

The integration domain, say for nth term in $\Psi_1(x,y)$, is the intersection of the region $y \le x_1 \le x_2 \le \ldots \le x_{2n+1}$ with the hyperplane $x+y-x_1+x_2-x_3+\ldots-x_{2n+1}=0$.

Lemma. *The series (1)–(6) are convergent.*

Consider, for example, the series for $\Phi_1(x,y)$. Let $Q = \max|q(x)|$.

$$|\int q(x_1)\bar{q}(x_2)q(x_3)\ldots\bar{q}(x_{2n})\Theta(y-x_1)\Theta_{12}\Theta_{23}\cdot\ldots\cdot\Theta_{2n-1,2n}$$
$$\delta(x-x_1+x_2-\ldots+x_{2n})\ dx_1\ldots dx_{2n}$$
$$\le Q\cdot\int |q(x_1)||q(x_2)|\ldots|q(x_{2n-1})|\Theta_{12}\Theta_{23}\ldots\Theta_{2n-1,2n}\ dx_1\ldots dx_{2n-1}$$
$$= \frac{Q}{(2n-1)!}\left(\int_{-\infty}^{\infty}|q(x)|dx\right)^{2n-1}.$$

Define $S(x) = \int \frac{b(k)}{a(k)} e^{2ikx}\frac{dk}{\pi}$, where

$$b(k) = \int b(x)e^{-2ikx}\ dx, \quad a(k) = \int a(x)e^{-2ikx}\ dx \tag{7}$$

Lemma.

1).For $q(x)$ such that $|1-a(k)|<1$ $S(x)$ is given by the convergent series

$$S(x) = \sum_{n=0}^{\infty}(-)^n\int q(x_1)\bar{q}(x_2)q(x_3)\ldots\bar{q}(x_{2n})q(x_{2n+1})$$
$$\Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54}\cdot\ldots\cdot\Theta_{2n-1,2n}\Theta_{2n+1,2n} \tag{8}$$
$$\cdot\delta(x-x_1+x_2-\ldots+x_{2n}-x_{2n+1})\ dx_1dx_2\ldots dx_{2n+1}$$

2).Consider $S(x)$,given by the formal series (8). We can prove the following relation:

$$\int S(x_1)a(x-x_1)dx_1 = b(x).$$

Here $a(x)$ and $b(x)$ are functionals in $q(x), \bar{q}(x)$, given by the series (1) and (2).

Indeed, let us collect all the terms of the same order in q, $\bar{q}$ in the convolution of the series (1) and (8)

$$\int S(y)a(x-y)dy = \int q(x_1)\delta(y-x_1)\delta(x-y)dx_1dy + \int q(x_1)\bar{q}(x_2)q(x_3)$$

$$(-\Theta_{12}\Theta_{32}\delta(y-x_1+x_2-x_3)\delta(x-y)+$$

$$\Theta_{32}\delta(y-x_1)\delta(x-y+x_2-x_3)\ dx_1dx_2dx_3dy$$

$$+\ldots+\int q(x_1)\bar{q}(x_2)\ldots q(x_{2n+1})(-)^n$$

$$(\Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54}\ldots\Theta_{2n-1,2n-2}\Theta_{2n-1,2n}\Theta_{2n+1,2n}$$

$$+\sum_{m=0}^{n-1}(-)^{m-n}\Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54}\ldots\Theta_{2m-1,2m}\Theta_{2m+1,2m}$$

$$(\Theta_{2m+2,2m+1}+\Theta_{2m+1,2m+2})\Theta_{2m+3,2m+2}$$

$$\cdot\,\Theta_{2m+4,2m+3}\cdot\ldots\cdot\Theta_{2n+1,2n})\,\delta(x-x_1+x_2-x_3+\ldots-x_{2n+1})\ dx_1\ldots dx_{2n}$$

$$=\sum_{n=0}^{\infty}\int q(x_1)\bar{q}(x_2)\ldots q(x_{2n+1})\Theta_{21}\Theta_{32}\Theta_{43}\cdot\ldots\cdot\Theta_{2n+1,2n}$$

$$\delta(x-x_1+x_2-x_3+\ldots-x_{2n+1})\ dx_1\ldots dx_{2n}$$

$$= b(x)$$

(We used $\Theta_{ij}+\Theta_{ji}=1$).

The convolutions of functionals (1)–(8) are again the functionals of the same type. There are certain relations for the convolutions:

$$\int \bar{a}(x_1)a(x+x_1)dx_1 = \delta(x) + \int \bar{b}(x_1)b(x+x_1)\ dx_1 \tag{9}$$

$$\int \Big(b(x_1)\Phi_2(x-x_1+y,y) - a(x_1)\bar{\Phi}_1(-x+x_1,y)\Big)\,dx_1 = -\Psi_2(x,y) \quad (10)$$

$$\int \Big(b(x_1)\Phi_1(x-x_1+y,y) - a(x_1)\bar{\Phi}_2(-x+x_1,y)\Big)\,dx_1 = -\Psi_1(x,y) \quad (11)$$

Let us prove (10). We have to collect all the terms of the same degree in q, $\bar{q}$ in the left-hand side and to compare with the right-hand side.

$$\int (\Phi_2(x-s+y,y)b(s) - \bar{\Phi}_1(-x+s,y)a(s))\ ds$$

$$= -\delta(x) + \int \bar{q}(x_1)q(x_2)\left(\Theta(y-x_1)\delta(x-s+x_1)\delta(s-x_2) - \Theta(y-x_1)\Theta_{12}\right.$$

$$\delta(x-s+x_1-x_2)d(s)\cdot -\Theta_{21}\delta(x-s)\delta(s+x_1-x_2))\,dx_1dx_2ds + \ldots$$

$$+ \int \bar{q}(x_1)q(x_2)\bar{q}(x_3)\ldots q(x_{2n})$$

$$\cdot\left(\sum_{m=0}^{n-1} \Theta(y-x_1)\Theta_{12}\Theta_{23}\Theta_{34}\cdot\ldots\cdot\Theta_{2m,2m+1}(\Theta_{2m+1,2m+2} + \Theta_{2m+2,2m+1})\right.$$

$$\Theta_{2m+3,2m+2}\Theta_{2m+4,2m+3}\cdot\ldots\cdot\Theta_{2n,2n-1} - \Theta_{21}\Theta_{32}\cdot\ldots\cdot\Theta_{2n,2n-1}$$

$$-\sum_{m=1} \Theta(y-x_1)\Theta_{12}\Theta_{23}\cdot\ldots\cdot\Theta_{2m-1,2m}(\Theta_{2m,2m+1} + \Theta_{2m+1,2m})$$

$$\Theta_{2m+2,2m+1}\cdot\Theta_{2m+3,2m+2}\cdot\ldots\cdot\Theta_{2n,2n-1}$$

$$\left.- \Theta(y-x_1)\Theta_{12}\Theta_{23}\cdot\ldots\cdot\Theta_{2n,2n-1}\right)\delta(x+x_1-x_2+\ldots-x_{2n})\ dx_1\ldots dx_{2n}$$

$$= -\delta(x) - \sum_{n=1}^{\infty}\int \bar{q}(x_1)q(x_2)\bar{q}(x_3)\ldots q(x_{2n})\Theta(x_1-y)\Theta_{21}\Theta_{32}\cdot\ldots\cdot\Theta_{2n,2n-1}$$

$$\delta(x+x_1-x_2+\ldots-x_{2n})dx_1\ldots dx_{2n} = -\Psi_2(x,y)$$

The proof of the other relations is similar.

In addition to convolution of two functionals, there is another operation for our functionals , namely inversion.It is the infinite-dimensional analoque of the inverse function. Consider the series (8)

$$
\begin{aligned}
S(x) = \sum_{i=0}^{\infty} q_{(2i+1)}(x) :=& \\
&= q(x) - \int q(x_1)\bar{q}(x_2)q(x_3)\Theta_{12}\Theta_{32}\delta(x - x_1 + x_2 - x_3) \\
&\quad + \int q(x_1)\bar{q}(x_2)q(x_3)\bar{q}(x_4)q(x_5)\Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54} \\
&\quad \delta(x - x_1 + x_2 - x_3 + x_4 - x_5) - \dots
\end{aligned}
$$

$S(x)$ is a formal series , its nth term to is a nonlocal analytic functional of $q(x)$, $\bar{q}(x)$ of degree $(2n+1)$ in $q, \bar{q}$. It can be inverted, namely, $q(x)$ can be expressed in terms of $S(x)$:

$$q(x) = S_{(1)}(x) + S_{(3)}(x) + S_{(5)}(x) + \dots$$

where $S_m(x)$ is a nonlocal analytic functional of $S(x), \bar{S}(x)$ of degree m in $S, \bar{S}$.

$$
\begin{aligned}
&S_{(1)}(x) = S(x) \\
&S_{(3)}(x) = \int S(x_1)\bar{S}(x_2)S(x_3)\delta(x - x_1 + x_2 - x_3)\Theta_{12}\Theta_{32}\ dx_1dx_2dx_3 \\
&S_{(5)}(x) = -\int S(x_1)\bar{S}(x_2)S(x_3)\bar{S}(x_4)S(x_5)\Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54} \\
&\qquad \delta(x - x_1 + x_2 - x_3 + x_4 - x_5)\ dx_1 \dots dx_5 \\
&\qquad + \int \Big(S_{(3)}^{(3)}(x_1)\bar{S}(x_2)S(x_3) + S(x_1) \\
&\qquad \bar{S}_{(3)}^{(3)}(x_2)S(x_3) + S(x_1)\bar{S}(x_2)S_{(3)}^{(3)}(x_3) \Big) \\
&\qquad \Theta_{12}\Theta_{32}\delta(x - x_1 + x_2 - x_3)\ dx_1dx_2dx_3 \\
&\quad = \int S(x_1)\bar{S}(x_2)S(x_3)\bar{S}(x_4)S(x_5)\delta(x - x_1 + x_2 - x_3 + x_4 - x_5)
\end{aligned}
$$

$$
\begin{aligned}
&(\, -\Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54} \\
&\quad + \Theta_{12}\Theta_{32} \qquad \Theta_{54}\Theta(x_1 - x_2 + x_3 - x_4) \\
&\qquad\qquad \Theta_{23}\Theta_{43} \qquad \Theta(x_1 - x_2 + x_3 - x_4)\Theta(x_5 - x_4 + x_3 - x_2) \\
&\quad + \Theta_{12} \qquad \Theta_{34}\Theta_{54}\Theta(x_5 - x_4 + x_3 - x_2)) \, dx_1 \dots dx_5 \\
&= \int S(x_1)\bar{S}(x_1)S(x_3)\bar{S}(x_4)S(x_5)\delta(x - x_1 + x_2 - x_3 + x_4 - x_5) \\
&\cdot\Theta_{12}\Theta(x_1 - x_2 + x_3 - x_4)\Theta_{54}\Theta(x_5 - x_4 + x_3 - x_2) \\
&(-\Theta_{32}\Theta_{34} + \Theta_{32} + \Theta_{23}\Theta_{43} + \Theta_{34}) \, dx_1 \dots dx_5 \\
&= \int S(x_1)\bar{S}(x_2)S(x_3)\bar{S}(x_4)S(x_5)\Theta_{12}\Theta(x_1 - x_2 + x_3 - x_4)\Theta_{54} \\
&\cdot\Theta(x_5 - x_4 + x_3 - x_2)\delta(x - x_1 + \dots - x_5) \, dx_1 \dots dx_5
\end{aligned}
$$

(Indeed, $-\Theta_{32}\Theta_{34} + \Theta_{32} + \Theta_{23}\Theta_{43} + \Theta_{34} = -\Theta_{32}\Theta_{34} + \Theta_{32}(\Theta_{34} + \Theta_{43}) + \Theta_{23}\Theta_{43} + \Theta_{34} = (\Theta_{23} + \Theta_{32})\Theta_{43} + \Theta_{34} = \Theta_{43} + \Theta_{34} = 1$.

Lemma. *Consider the nonlocal analytic functionals of $\{S(x)\}$, given by formal series*

$$
\begin{aligned}
\tilde{q}(x) = S(x) + \sum_{n=1}^{\infty} \int &\Big(S(x_1)\bar{S}(x_2)S(x_3) \dots \bar{S}(x_{2n})S(x_{2n+1}) \\
&\Theta(x_1 - x_2)\Theta(x_1 - x_2 + x_3 - x_4) \dots \Theta(x_1 - x_2 + \dots - x_{2n}) \\
&\Theta(x_{2n+1} - x_{2n})\Theta(x_{2n+1} - x_{2n} + x_{2n-1} - x_{2n-2}) \dots \Theta(x_{2n+1} - x_{2n} + \dots - x_2) \\
&\delta(x - x_1 + x_2 - \dots + x_{2n} - x_{2n+1}) \, dx_1 \dots dx_{2n+1}
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\tilde{\Phi}_1(x, y) = \delta(x) + \sum_{n=1}^{\infty} \int &\Big(S(x_1)\bar{S}(x_2)S(x_3) \dots \bar{S}(x_{2n}) \\
&\cdot\Theta(y - x_1)\Theta(y - x_1 + x_2 - x_3) \dots \Theta(y - x_1 + \dots - x_{2n-1}) \\
&\cdot\Theta(x_1 - x_2)\Theta(x_1 - x_2 + x_3 - x_4) \cdot \ldots \cdot \Theta(x_1 - x_2 + \dots - x_{2n}) \\
&\delta(x - x_1 + x_2 - x_3 + \dots - x_{2n+1} + x_{2n}) \, dx_1 \dots dx_{2n}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\tilde{\Phi}_2(x,y) = \sum_{n=0}^{\infty} \int & \bar{S}(x_1) S(x_2) \bar{S}(x_3) \ldots S(x_{2n}) \bar{S}(x_{2n+1}) \\
& \cdot \Theta(y - x_1) \cdot \Theta(y - x_1 + x_2 - x_3) \cdot \ldots \cdot \Theta(y - x_1 + x_2 - x_3 + \ldots - x_{2n-1}) \\
& \cdot \Theta(x_1 - x_2) \Theta(x_1 - x_2 + x_3 - x_4) \cdot \ldots \cdot \Theta(x_1 - x_2 + \ldots - x_{2n}) \\
& \delta(x - y + x_1 - x_2 + \ldots - x_{2n} + x_{2n+1}) \, dx_1 \ldots dx_{2n+1}
\end{aligned} \tag{14}
$$

Let us substitute in these series $S(x)$ as formal series in $\{q(x)\}$, (8). The result of such substitution would be formal series in $\{q(x)\}$, and , moreover,

$$
\begin{aligned}
\tilde{q}(x) &= q(x) \\
\tilde{\Phi}_1(x,y) &= \Phi_1(x,y) \\
\tilde{\Phi}_2(x,y) &= \Phi_2(x,y)
\end{aligned}
$$

as formal series in $q(x)$, with $\Phi_1(x,y)$ and $\Phi_2(x,y)$ given by (3) , (4).

The series (9) could be used to get the solution of a nonlinear equation. In order to see this, let us rewrite (9) in terms of $S(k) = \int_{-\infty}^{\infty} S(x) e^{-2ikx} dx$ and $\bar{S} = (k) \int_{-\infty}^{\infty} \bar{S}(x) e^{2ikx} dx$:

$$
\begin{aligned}
q(x) = 2 \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} & \frac{S(k_1) \bar{S}(k_2) \ldots S(k_{2n+1})}{(k_2 - k_1 + i0)(k_3 - k_2 - i0)(k_4 - k_3 + i0) \ldots (k_{2n+1} - k_{2n} - i0)} \\
& \cdot \exp(2i(k_1 - k_2 + k_3 - \ldots + k_{2n+1})x) \, \frac{dk_1 \ldots dk_{2n+1}}{(2\pi)^{2n+1}}
\end{aligned} \tag{15}
$$

The kernels $\dfrac{1}{k \mp i0} = 2i \int_{-\infty}^{\infty} \Theta(x) exp(\mp 2ikx)$ appeared as the Fourier transform of the Heaviside function kernels.

The series (15) has the following property: a polynomial in $q(x)$, $\bar{q}(x)$ and their derivatives can also be written in the form (15) but with some polynomial in $\{k\}$ in the numerator:

$$\frac{d^n q(x)}{dx^n} = 2\cdot(2i)^n \sum_{n=0}^{\infty}$$
$$\int \frac{S(k_1)\bar{S}(k_2)\dots S(k_{2n+1})(k_1-k_2+\dots-k_{2n}+k_{2n+1})^n}{(k_2-k_1+i0)(k_3-k_2-i0)(k_4-k_3+i0)\dots(k_{2n+1}-k_{2n}-i0)}$$
$$\cdot\exp(2i(k_1-k_2+k_3-\dots+k_{2n+1})x)\ \frac{dk_1\dots dk_{2n+1}}{(2\pi)^{2n+1}}$$

$$q(x)\bar{q}(x) = 4\sum_{n=1}^{\infty}$$
$$\int \frac{S(k_1)\bar{S}(k_2)S(k_3)\dots\bar{S}(k_{2n})(-k_1+k_2-\dots+k_{2n})}{(k_2-k_1+i0)(k_3-k_2-i0)(k_4-k_3+i0)\dots(k_{2n-1}-k_{2n-2}-i0)(k_{2n}-k_{2n-1}+i0)}$$
$$\cdot\exp(2i(k_1-k_2+\dots-k_{2n})x)\ \frac{dk_1\dots dk_{2n}}{(2\pi)^{2n}}$$

$$q(x)\bar{q}(x)q(x) = 4\sum_{n=1}^{\infty}$$
$$\int \frac{S(k_1)\bar{S}(k_2)S(k_3)\dots\bar{S}(k_{2n})S(k_{2n+1})}{(k_2-k_1+i0)(k_3-k_2-i0)\dots(k_{2n}-k_{2n-1}-i0)(k_{2n+1}-k_{2n}+i0)}$$
$$\cdot\left(-(k_1-k_2+\dots-k_{2n+1})^2-\sum_{p=1}^{n}{k_{2p}}^2+\sum_{p=0}^{n}{k_{2p+1}}^2\right)$$
$$\exp(2i(k_1-k_2+\dots-k_{2n}+k_{2n+1}))\ \frac{dk_1\dots dk_{2n+1}}{(2\pi)^{2n+1}}$$

$$q^{m+1}(x)\bar{q}^m(x) = 2^{2m+1}\sum_{n=m}^{\infty}$$
$$\Bigg(\int \frac{S(k_1)\bar{S}(k_2)S(k_3)\dots\bar{S}(k_{2n})S(k_{2n+1})}{(k_2-k_1-i0)(k_3-k_2+i0)\dots(k_{2n}-k_{2n-1}-i0)(k_{2n+1}-k_{2n}+i0)}$$
$$\cdot\sum_{1\le p_1<p_2<\dots<p_m\le n}(-k_1+k_2-\dots+k_{2p_1})$$
$$(k_{2p_1+1}-k_{2p_1})(-k_{2p_1+1}+k_{2p_1+2}-\dots+k_{2p_2})(k_{2p_2+1}-k_{2p_2})\cdot\dots$$
$$\cdot(-k_{2p_m+1}+k_{2p_m+2}-\dots+k_{2p_m})(k_{2p_m+1}-k_{2p_m})\ \frac{dk_1\dots dk_{2n+1}}{(2\pi)^{2n+1}} \tag{16}$$

The fact that both differentiation of $q(x)$ and nonlinearity under the "nonlinear Fourier transformation" (15) has the same effect, namely, some polynomial in $\{k\}$ appears in the numerator, can be used to solve nonlinear equations.

Theorem 3.1 (solution of the NLS Equation).

Consider the Cauchy problem for the NLS equation:

$$
\begin{aligned}
& i\frac{\partial}{\partial t}q(x,t)+\frac{\partial^2}{\partial x^2}q(x,t)-2\lambda|q|^2q=0, \quad \lambda=\begin{cases} \;\;1, & \text{defocusing case} \\ -1, & \text{focusing case}\end{cases} \\
& q(x,0)=q(x)
\end{aligned}
\tag{17}
$$

(Solution in formal series).

The solution of Cauchy problem is given by

$$
\begin{aligned}
q(x,t)=2\sum_{n=0}^{\infty}\lambda^n\int & \frac{S(k_1,t)\bar{S}(k_2,t)\ldots S(k_{2n+1},t)}{(k_2-k_1+i0)(k_3-k_2-i0)\ldots(k_4-k_3+i0)(k_{2n+1}-k_{2n}-i0)} \\
& \exp(2i(k_1-k_2+\ldots-k_{2n}+k_{2n+1}) \\
& \frac{dk_1\ldots dk_{2n+1}}{(2\pi)^{2n+1}}
\end{aligned}
\tag{18}
$$

$$S(k,t)=e^{-4ik^2t}S(k)$$

$$
\begin{aligned}
S(k)=\sum_{n=0}^{\infty}(-)^n\lambda^n & \int_{-\infty}^{\infty}q(x_1)\bar{q}(x_2)q(x_3)\ldots\bar{q}(x_{2n})q(x_{2n+1}) \\
& \Theta_{12}\Theta_{32}\Theta_{34}\Theta_{54}\ldots\Theta_{2n-1,2n}\Theta_{2n+1,2n} \\
& \exp(-2ikx)\delta(x-x_1+x_2-\ldots+x_{2n}-x_{2n+1})\; dxdx_1dx_2\ldots dx_{2n+1}
\end{aligned}
\tag{19}
$$

(Solution in convergent series, defocusing case $\lambda=1$).

$$q(x,t)=\int S(x-y,t)\left(\delta(y)-\frac{D_{x,t}\begin{pmatrix}y\\0\end{pmatrix}}{D_{x,t}}\right)dy$$

$$S(x,t) = \int \frac{b(k)}{a(k)} \exp(2ikx - 4ik^2 t) \frac{dk}{\pi}, \ b(k) \text{ and } a(k) \text{ are defined by (1) (2)}$$

$$D_{x,t}\binom{y}{y_0} = -\Theta(-y) K_{x,t}(y, y_0) + \sum_{n=1}^{\infty} \frac{(-)^{n+1}}{n!}$$

$$\Theta(-y) \int \Theta(-y_1)\Theta(-y_2)\ldots\Theta(-y_n)$$

$$\cdot \begin{vmatrix} K_{x,t}(y,y_0) & K_{x,t}(y,y_1) & \ldots & K_{x,t}(y,y_n) \\ K_{x,t}(y_1,y_0) & K_{x,t}(y_1,y_1) & \ldots & K_{x,t}(y_1,y_n) \\ & \ldots & & \\ K_{x,t}(y_n,y_0) & K_{x,t}(y_n,y_1) & \ldots & K_{x,t}(y_n,y_n) \end{vmatrix} dy_1 \ldots dy_n$$

$$D_{x,t} = 1 + \sum_{n=1}^{\infty} \frac{(-)^n}{n!} \int \Theta(-y_1)\Theta(-y_2)\ldots\Theta(-y_n) \ |K_{x,t}(y_i,y_j)|_{i,j} \ dy_1 \ldots dy_n$$

$$K_{x,t}(y_1,y_2) = \int \Theta(-y)\bar{S}(x - y_1 - y) S(x - y - y_2, t) \ dy$$

The solution is defined for the class of initial data $q(x)$ such that the Fredholm determinant $D_{x,t}$ and the first minor $D_{x,t}\binom{y}{y_0}$ are convergent. It can be proved that the determinant is not zero.

Proof (formal series).

1) We compute $\frac{\partial^2}{\partial x^2} q(x,t)$ and $|q|^2 q$ in the same way as before, see (16) ;

$$i\frac{\partial}{\partial t} q(x,t) + \frac{\partial^2}{\partial x^2} q(x,t) - 2|q|^2 q$$

$$= 2\int (i\frac{\partial}{\partial t} - 4{k_1}^2) S(k,t) e^{2ik_1 x} \ \frac{dk_1}{2\pi}$$

$$+ \sum_{n=1}^{\infty} 2\lambda^n (i\frac{\partial}{\partial t} - 4({k_1}^2 - {k_2}^2 + {k_3}^2 - \ldots + {k_{2n+1}}^2))$$

$$\frac{S(k_1,t)\bar{S}(k_2,t)\ldots S(k_{2n+1},t)}{(k_2 - k_1 + i0)(k_3 - k_2 - i0)\ldots(k_{2n+1} - k_{2n} - i0)}$$

$$\cdot \exp(2i(k_1 - k_2 + k_3 - \ldots + k_{2n+1})x) \ \frac{dk_1 \ldots dk_{2n+1}}{(2\pi)^{2n+1}} = 0$$

for $S(k,t) = e^{-4ik^2 t} S(k,0)$.

2) The substitution of $S(k,0)=S(k)$ as series in $\{q(x)\}$ (see (19)) into (18) gives $q(x,0)=q(x)$ (see Lemma) .

4 Nonlinear transformations for Davey-Stewardson equation

Let $q(x,y)$ be a complex-valued C^∞ function on the plane $\mathbb{R}^2$, with fast decrease as $|x|^2+|y|^2\to\infty$.

We construct the following nonlocal analytic functionals of $\{q\}$:

$$\alpha(k,\bar k)=\sum_{n=0}^{\infty} \int \frac{q(z_1,\bar z_1)\bar q(z_2,\bar z_2)q(z_3,\bar z_3)\dots\bar q(z_{2n},\bar z_{2n})q(z_{2n+1},\bar z_{2n+1})}{(\bar z_1-\bar z_2)(z_2-z_3)(\bar z_3-\bar z_4)\dots(z_{2k}-z_{2k+1})} \cdot\exp\Big(\bar k(\bar z_1-\bar z_2+\bar z_3-\ldots+\bar z_{2k+1})-k(z_1-z_2+z_3-\ldots+z_{2k+1})\Big) \frac{d^2z_1d^2z_2\dots d^2z_{2n+1}}{(2\pi)^{2n+1}} \tag{1}$$

$$\mu_1(k,\bar k;z,\bar z)=1+\sum_{n=1}^{\infty} \int \frac{q(z_1,\bar z_1)\bar q(z_2,\bar z_2)\dots\bar q(z_{2n-1},\bar z_{2n-1})\bar q(z_{2n},\bar z_{2n})}{(z-z_1)(\bar z_1-\bar z_2)(z_2-z_3)\dots(\bar z_{2n-1}-\bar z_{2n})} \cdot\exp\Big(\bar k(\bar z_1-\bar z_2+\bar z_3-\ldots+\bar z_{2n+1}-\bar z_{2n})-k(z_1-z_2+z_3-\ldots+z_{2n+1}-z_{2n})\Big) \frac{d^2z_1\dots d^2z_{2n}}{(2\pi)^{2n}} \tag{2}$$

$$\mu_2(k,\bar{k};z,\bar{z}) = \sum_{n=0}^{\infty}$$
$$\int \frac{\bar{q}(z_1,\bar{z}_1)q(z_2,\bar{z}_2)\dots\bar{q}(z_{2n+1},\bar{z}_{2n+1})}{(\bar{z}-\bar{z}_1)(z_1-z_2)(\bar{z}_2-\bar{z}_3)\dots(z_{2n-1}-z_{2n})(\bar{z}_{2n}-\bar{z}_{2n+1})}$$
$$\cdot \exp\Big(\bar{k}(\bar{z}-\bar{z}_1+\bar{z}_2-\ldots+\bar{z}_{2n}-\bar{z}_{2n+1}) - k(z-z_1+z_2-\ldots+z_{2n}-z_{2n+1})\Big)$$
$$\frac{d^2z_1\dots d^2z_{2n}}{(2\pi)^{2n+1}} \tag{3}$$

where $z = x_1 + ix_2$, $\bar{z} = x_1 - ix_2$, $k = k_1 + ik_2$, $\bar{k} = k_1 - ik_2$, $d^2z := \frac{i}{2}dzd\bar{z}$.

The series are convergent for some class of functions q. We will not investigate convergence; we will work with these functionals as with formal series. Each term of these series is an integral, involving homogeneous generalized functions $\frac{1}{z}$ as kernels ([1]).

There are the following relations for the functionals (1)–(3):

$$\frac{\partial\mu_1}{\partial\bar{z}}(k,\bar{k},z,\bar{z}) = \frac{1}{2}q(z,\bar{z})\mu_2(k,\bar{k},z,\bar{z}).$$
$$\frac{\partial\mu_2}{\partial\bar{z}}(k,\bar{k},z,\bar{z}) = k\mu_2(k,\bar{k},z,\bar{z}) + \frac{1}{2}q(z,\bar{z})\mu_1(k,\bar{k},z,\bar{z}).$$
$$\frac{\partial\mu_1}{\partial\bar{k}} = e^{\bar{k}\bar{z}-kz}\bar{\alpha}\bar{\mu}_2.$$
$$\frac{\partial\mu_2}{\partial\bar{k}} = e^{\bar{k}\bar{z}-kz}\bar{\alpha}\bar{\mu}_1.$$

The series (1) could be inverted, that is, $q(z,\bar{z})$ could be written as a functional of $\{\alpha(k,\bar{k})\}$:

$$\alpha(k,\bar{k}) = \int q(z_1,\bar{z}_1)\exp(\bar{k}\bar{z}_1 - kz_1)\frac{d^2z_1}{2\pi} + \int\Bigg(\frac{q(z_1,\bar{z}_1)\bar{q}(z_2,\bar{z}_2)q(z_3,\bar{z}_3)}{(z_1-z_2)(\bar{z}_2-\bar{z}_3)}$$
$$\exp(\bar{k}(\bar{z}_1-\bar{z}_2+\bar{z}_3) - k(z_1-z_2+z_3))\quad\Bigg)\frac{d^2z_1d^2z_2d^2z_3}{(2\pi)^3} + \ldots$$
$$q(z,\bar{z}) = \alpha_{(1)}(z,\bar{z}) + \alpha_{(3)}(z,\bar{z}) + \ldots$$
$$\alpha_{(1)}(z,\bar{z}) = -2\int\alpha(k_1,\bar{k}_1)\exp(-\bar{k},\bar{z}+k\bar{z})\ \frac{d^2k}{\pi}$$

$$\alpha_{(3)}(z,\bar z) = -\frac{2}{\pi^7}\int \frac{\alpha(k_1,\bar k_1)\bar\alpha(k_2,\bar k_2)\alpha(k_3,\bar k_3)}{(z_1-z_2)(\bar z_2-\bar z_3)}$$

$$\exp\Big((-\bar k_1\bar z_1+\bar k_2\bar z_2-\bar k_3 z_3+\bar k(\bar z_1-\bar z_2+\bar z_3)-\bar k\bar z)$$

$$d^2z_1d^2z_2d^2z_3d^2k_1d^2k_2d^2k_3d^2k$$

$$= -\frac{2}{\pi^3}\int \frac{\alpha(k_1,\bar k_1)\bar\alpha(k_2,\bar k_2)\bar\alpha(k_3,\bar k_3)}{(\bar k-\bar k_1)(k_2-k_1)}\ \delta(k-k_1+k_2-k_3)$$

$$\exp(-\bar k\bar z+kz)d^2k_1d^2k_2d^2k_3d^2k_4$$

$$= -2\int \frac{\alpha(k_1,\bar k_1)\bar\alpha(k_2,\bar k_2)\alpha(k_3,\bar k_3)}{(k_2-k_1)(\bar k_3-\bar k_2)}$$

$$\exp(\bar z(-\bar k_1+\bar k_2-\bar k_3)-z(-k_1+k_2-k_3))\ \frac{d^2k_1d^2k_2d^2k_3}{\pi^3}$$

Lemma. *Consider the following functionals of* $\{\alpha(k,\bar k)\}$ *:*

$$\tilde\mu_1(k,\bar k,z,\bar z) = 1+\sum_{n=1}^{\infty}$$

$$\int \frac{\alpha(k_1,\bar k_1)\bar\alpha(k_2,\bar k_2)\ldots\alpha(k_{2n-1},k_{2n-1})\bar\alpha(k_{2n},k_{2n})}{(\bar k_2-\bar k_1)(k_3-k_2)(\bar k_4-\bar k_3)\ldots(k_{2n-1},k_{2n-2})(\bar k_{2n}-\bar k_{2n-1})(k-k_{2n})}$$

$$\cdot\exp(\bar z(-\bar k_1+\bar k_2-\bar k_3+\ldots-\bar k_{2n-1}+\bar k_{2n})$$

$$-z(-k_1+k_2-k_3+\ldots-k_{2n-1}+k_{2n}))\ \frac{d^2k_1\ldots d^2k_n}{\pi^{2n}}, \tag{4}$$

$$\tilde{\mu}_2(k,\bar{k},z,\bar{z}) = \sum_{n=0}^{\infty}$$

$$\int \frac{\bar{\alpha}(k_1,\bar{k}_1)\alpha(k_2,\bar{k}_2)\ldots\bar{\alpha}(k_{2n-1},\bar{k}_{2n-1})\alpha(k_{2n},k_{2n})\bar{\alpha}(k_{2n+1},\bar{k}_{2n+1})}{(k_2-k_1)(\bar{k}_3-\bar{k}_2)(k_4-k_3)\ldots(\bar{k}_{2n-1}-\bar{k}_{2n-2})(k_{2n}-k_{2n-1})(\bar{k}_{2n+1}-\bar{k}_{2n})(k-k_{2n+1})}$$

$$\exp(\bar{z}(\bar{k}_1-\bar{k}_2+\ldots+\bar{k}_{2n+1})-z(k_1-k_2+k_3-\ldots+k_{2n-1})),$$

$$\frac{d^2k_1\ldots d^2k_{2n+1}}{\pi^{2n+1}} \tag{5}$$

$$\tilde{q}(z,\bar{z}) = -2\sum_{n=0}^{\infty}$$

$$\int \frac{\alpha(k_1,k_1)\bar{\alpha}(k_2,k_2)\ldots\alpha(k_{2n+1},\bar{k}_{2n+1})}{(k_2-k_1)(\bar{k}_3-\bar{k}_2)(k_4-k_3)\ldots(\bar{k}_{2n+1}-\bar{k}_{2n})}$$

$$\cdot\exp(\bar{z}(-\bar{k}_1+\bar{k}_2-\ldots+\bar{k}_{2n}-\bar{k}_{2n+1})-z(-k_1+k_2-\ldots+k_{2n}-k_{2n+1}))$$

$$\frac{d^2k_1\ldots d^2k_{2n+1}}{\pi^{2n+1}}. \tag{6}$$

We can substitute in (4)–(6) $\alpha(k,\bar{k})$ *as a functional in* $q(z,\bar{z})$*, given by* (1)*. As a result of this substitution we will obtain functionals in* $\{q(z,\bar{z})\}$ *,given by formal series. Moreover,*

$$\tilde{\mu}_1(k,\bar{k},z,\bar{z}) = \tilde{\mu}_1(k,\bar{k},z,\bar{z})$$

$$\tilde{\mu}_2(k,\bar{k},z,\bar{z}) = \tilde{\mu}_2(k,\bar{k},z,\bar{z})$$

$$\tilde{q}(z,\bar{z}) = q(z,\bar{z})$$

Theorem 4.1 (Solution of the Davey-Stewartson equation-II)

Consider the DS equation

$$i\frac{\partial}{\partial t}q(z,\bar z,t) = -(\partial^2+\bar\partial^2)q(z,\bar z,t)$$
$$+\frac{1}{2}q(z,\bar z,t)(\bar\partial^{-1}\partial+\partial^{-1}\bar\partial)(|q(z,\bar z,t)|^2)$$
$$q(z,\bar z,0)=q(z,\bar z)$$

(here $\partial=\dfrac{\partial}{\partial z}$, $\bar\partial=\dfrac{\partial}{\partial\bar z}$, $\bar\partial^{-1}f(z,\bar z)=\dfrac{1}{\pi}\displaystyle\int\frac{f(z^1,\bar z^1)}{z-z^1}\,d^2z^1$.

The solution is

$$q(z,\bar z,t) = -2\sum_{n=0}^{\infty}$$
$$\int\frac{\alpha(k_1,\bar k_1,t)\bar\alpha(k_2,k_2,t)\ldots\alpha(k_{2n+1},\bar k_{2n+1},t)}{(k_1-k_2)(\bar k_2-\bar k_3)(k_3-k_4)\ldots(\bar k_{2n}-\bar k_{2n+1})}$$
$$\cdot\exp(\bar z(-\bar k_1+\bar k_2-\ldots+\bar k_{2n}-\bar k_{2n+1})-z(-k_1+k_2-\ldots+k_{2n}-k_{2n+1}))$$
$$\frac{d^2k_1\ldots d^2k_n}{\pi^{2n+1}} \tag{7}$$

$$\alpha(k,\bar k)=\sum_{n=0}^{\infty}$$
$$\int\frac{q(z_1,\bar z_1)\bar q(z_2,\bar z_2)q(z_3,\bar z_3)\ldots\bar q(z_{2n},\bar z_{2n})q(z_{2n+1},\bar z_{2n+1})}{(\bar z_1-\bar z_2)(z_2-z_3)(\bar z_3-\bar z_4)\ldots(z_{2k}-z_{2k+1})} \tag{8}$$
$$\cdot\exp(\bar k(-\bar z_1-\bar z_2+\bar z_3-\ldots+\bar z_{2k+1}-k(z_1+z_2+z_3-\ldots+z_{2n+1}))$$
$$\frac{d^2z_1d^2z_2\ldots d^2z_{2n+1}}{(2\pi)^{2n+1}}$$

$$\alpha(k,\bar k,t)=\alpha(k,\bar k)e^{i(k^2+\bar k^2)t}.$$

Proof.

1) Let us compute $q\bar{\partial}^{-1}\partial(|q|^2)$ for $q(z,\bar{z},t)$ given by (6):

$$q\bar{\partial}^{-1}\partial(|q|^2) = -8\sum_{n=1}^{\infty}$$
$$\int \frac{\alpha(k_1,\bar{k}_1,t)\bar{\alpha}(k_2,\bar{k}_2,t)\ldots\alpha(k_{2n+1},\bar{k}_{2n+1},t)}{(k_1-k_2)(\bar{k}_2-\bar{k}_3)(k_3-k_4)\ldots(\bar{k}_{2n}-\bar{k}_{2n+1})}$$
$$\cdot\exp(\bar{z}(-\bar{k}_1+\bar{k}_2-\ldots+\bar{k}_{2n}-\bar{k}_{2n+1})-z(-k_1+k_2-\ldots+k_{2n}-k_{2n+1}))$$

$$\cdot\sum_{\substack{m_1,m_2\geq 0\\ m_1+m_2+1\leq n}}(k_{2m_1+1}-k_{2m_1+2})(\bar{k}_{2m_1+2m_2+2}-\bar{k}_{2m_1+2m_2+3})$$
$$\frac{k_{2m_1+2}-k_{2m_1+3}+\ldots-k_{2n+1}}{\bar{k}_{2m_1+2}-\bar{k}_{2m_1+3}+\ldots-\bar{k}_{2n+1}}$$
$$\frac{d^2k_1\ldots d^2k_n}{\pi^{2n+1}}$$

In the sum over m_1, m_2 let us sum over m_2 first, then the second multiplier and the denumerator cancels:

$$\sum_{\substack{m_1,m_2\geq 0\\ m_1+m_2+1\leq n}}(k_{2m_1+1}-k_{2m_1+2})(\bar{k}_{2m_1+2m_2+2}-\bar{k}_{2m_1+2m_2+3})$$
$$\frac{k_{2m_1+2}-k_{2m_1+3}+\ldots-k_{2n+1}}{\bar{k}_{2m_1+2}-\bar{k}_{2m_1+3}+\ldots-\bar{k}_{2n+1}}$$
$$=\sum_{m_1\geq 0}^{n-1}(k_{2m_1+1}-k_{2m_1+2})(\bar{k}_{2m_1+2}-\bar{k}_{2m_1+3}\bar{k}_{2m_1+4}-\bar{k}_{2m_1+5}+\ldots+\bar{k}_{2n}-\bar{k}_{2n+1})$$
$$\cdot\frac{(k_{2m_1+2}-k_{2m_1+3}+\ldots-k_{2n+1})}{\bar{k}_{2m_1+2}-\bar{k}_{2m_1+3}+\ldots-\bar{k}_{2n+1}}$$
$$=\sum_{m_1=0}^{n-1}(k_{2m_1+1}-k_{2m_1+2})(k_{2m_1+2}-k_{2m_1+3}+\ldots-k_{2n+1})$$
$$=\frac{1}{2}\left(-(k_1-k_2+k_3-\ldots+k_{2n+1})^2-\sum_{p=1}^{n}{k_{2p}}^2+\sum_{p=0}^{n}{k_{2p+1}}^2\right)$$

Let us substitute $q(z,\bar{z},t)$ as a functional of $\{\alpha(k,\bar{k},t)\}$ in the equation

$$\int (i\partial_t + {k_1}^2 + \bar{k}_1^2)\alpha(k_1,k_1,t)\exp(-\bar{z}_1,\bar{k}_1 + z_1k_1)\frac{d^2k_1}{\pi}$$

$$+\sum_{n=1}^{\infty}\int\left(i\partial_t + \sum_{p=0}^{n}(k_{2p+1}^2 + \bar{k}_{2p+1}^2) - \sum_{p=1}^{n}({k_{2p}}^2 + \bar{k}_{2p}^2)\right)$$

$$\frac{\alpha(k,\bar{k},t)\bar{\alpha}(k_2,\bar{k}_2,t)\ldots\alpha(k_{2n+1},\bar{k}_{2n+1},t)}{(k_1-k_2)(\bar{k}_2-\bar{k}_3)(k_3-k_4)\ldots(\bar{k}_{2n}-\bar{k}_{2n+1})}$$

$$\exp(\bar{z}(-\bar{k}_1+\bar{k}_2-\ldots+\bar{k}_{2n}-\bar{k}_{2n+1}) - z(-k_1+k_2-\ldots+k_{2n}-k_{2n+1}))$$

$$\frac{d^2k_1\ldots d^2k_{2n+1}}{\pi^{2n+1}} = 0$$

if $\alpha(k,\bar{k},t) = \alpha(k,\bar{k})e^{i(k^2+\bar{k}^2)t}$

2) Substitution of $\alpha(k,\bar{k},0) = \alpha(k,\bar{k})$, where $\alpha(k,\bar{k})$ is given by (8), into (7) gives $q(z,\bar{z},0) = q(z,\bar{z})$.

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